The scattering matrix

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Scattering matrix in $\mathbb{R}$

Recall the scattering matrix in $\mathbb{R}$. At frequency λ and $|x| \gg 1$, the solution of

$$(P_V - \lambda^2)u = 0$$

is

$$u(x) = b_{\operatorname{sgn} x} e^{-i\lambda|x|} + a_{\operatorname{sgn} x} e^{i\lambda|x|}$$

and the scattering matrix $S(\lambda)$ is defined by

$$S(\lambda)(b_+, b_-) = (a_-, a_+),$$

i.e. $S(\lambda)$ maps the amplitudes of incoming waves to amplitudes of outgoing waves.

Scattering matrix in $\mathbb{R}$

In other words, for $\theta, \omega \in \mathbb{S}^0 = \{+, -\}$ and δ the delta mass on $\mathbb{S}^0$,

$$(S(\lambda)\delta_\omega)(\theta) = \delta_\omega(\theta) + v_\theta^\omega(\lambda)$$

where

$$v^\pm_{\operatorname{sgn} x}(\lambda) = -e^{-i\lambda|x|} R_V(\lambda)(Ve^{\pm i\lambda \bullet})(x)$$

are the reflection and transmission coefficients, which don't depend on x since they expand as a Wronskian. Thus $S(\lambda)$ acts on $\ell^2(\mathbb{S}^0) \cong \mathbb{C}^2$.

We generalize this to nD, $n \geq 3$ odd, by defining $S(\lambda)$ to be an operator on $L^2(\mathbb{S}^{n-1})$.

Scattering matrix in $\mathbb{R}^n$

Let $\lambda \in \mathbb{R} \setminus 0$. Let

$$u(x, \lambda, \omega) = -R_V(\lambda)(Ve^{-i\lambda\langle \bullet, \omega\rangle})(x)$$

be the outgoing part of the eigenfunction $w(x, \lambda, \omega) = e^{-i\lambda\langle x, \omega\rangle} + u(x, \lambda, \omega)$ of P_V of frequency λ. Here $\omega \in \mathbb{S}^{n-1}$. Recall from James' talk:

Lemma (outgoing solutions asymptotics)

Let f be a compactly supported distribution, $(P_V - \lambda^2)u = f$, $V \in L^\infty_c(\mathbb{R}^n \to \mathbb{R})$, $n \geq 3$ odd, $\lambda \in \mathbb{R} \setminus 0$. Then u is outgoing if and only if there is a function b such that

$$u(r\theta, \lambda, \omega) = c_n^-(\lambda r)^{-(n-1)/2} e^{i\lambda r} b(\lambda, \theta, \omega) + O(r^{-(n+1)/2})$$

We call $b(\lambda, \theta, \omega)$ the *scattering amplitude* of u in the directions θ, ω and at frequency λ. The point is that $b(\lambda, \theta, \omega)$ does not depend on r.

Scattering matrix in $\mathbb{R}^n$

Recall from James' talk:

Lemma (stationary phase on the sphere)

If $\lambda \in \mathbb{R} \setminus 0$, then as $r \to \infty$,

$$e^{-i\lambda\langle r\theta,\omega\rangle} \sim (r\lambda)^{-(n-1)/2}\left(c_n^+ e^{-ir\lambda}\delta_\omega(\theta) + c_n^- e^{ir\lambda}\delta_{-\omega}(\theta)\right).$$

Plugging everything in, we see that

$$w(r\theta,\lambda,\omega) \sim \frac{c_n^+}{(\lambda r)^{\frac{n-1}{2}}}\left(e^{-i\lambda r}\delta_\omega(\theta) + e^{i\lambda r} i^{1-n}(\delta_{-\omega}(\theta) + b(\lambda,\theta,\omega)\right)$$

where the $e^{-i\lambda r}$ term is incoming and the $e^{i\lambda r}$ term is outgoing. Define the *absolute scattering matrix* $S_{abs}(\lambda)\delta_\omega(\theta) = i^{1-n}(\delta_{-\omega}(\theta) + b(\lambda,\theta,\omega)))$ to map the incoming amplitude to the outgoing amplitude. This is annoying to write so we normalize the *scattering matrix* as

$$S(\lambda)\delta_\omega(\theta) = \delta_\omega(\theta) + b(\lambda,\theta,-\omega).$$

Scattering matrix action on L^2

Theorem

Let $n \geq 3$ odd, $V \in L^\infty_c(\mathbb{R}^n \to \mathbb{R})$, λ not a resonance of V. Then $S(\lambda)$ acts on $L^2(\mathbb{S}^{n-1})$ by

$$S(\lambda)f(\theta) = f(\theta) + \int_{\mathbb{S}^{n-1}} b(\lambda, \theta, -\omega) f(\omega) \, d\omega.$$

To prove this, fix ρ and recall from Izak's talk that $E_\rho(\lambda, \omega, x) = \rho(x) e^{-i\lambda\langle x, \omega\rangle}$ is an integral operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$.

Lemma

With hypotheses as above, $a_n = (2\pi)^{1-n}/2$, and

$$A(\lambda) = a_n \lambda^{n-2} E_\rho(\lambda)(1 + VR_0(\lambda)\rho)^{-1} V E_\rho(\overline{\lambda})^*,$$

one has $S = 1 + A$.

This lemma shows that the definition of $S(\lambda)$ extends to $\lambda \in \mathbb{C}$.

Scattering matrix action on L^2

Proof

Recall from Yonah's talk that if λ is not a resonance, then $1 + VR_0(\lambda)\rho$ is invertible on $L^2(\mathbb{R}^n)$. Since $E_\rho(\lambda)$ conjugates $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{S}^{n-1})$ and assuming the lemma we have

$$S(\lambda) = 1 + a_n \lambda^{n-2} E_\rho(\lambda)(1 + VR_0(\lambda)\rho)^{-1} V E_\rho(\overline{\lambda})^*,$$

so $S(\lambda)$ is a bounded operator on $L^2(\mathbb{R}^n)$. This justifies the computation

$$\begin{aligned} S(\lambda)f(\theta) &= S(\lambda) \int_{\mathbb{S}^{n-1}} \delta_\omega(\theta) f(\omega)\ d\omega \\ &= \int_{\mathbb{S}^{n-1}} (\delta_\omega(\theta) + b(\lambda, \theta, -\omega)) f(\omega)\ d\omega \\ &= f(\theta) + \int_{\mathbb{S}^{n-1}} b(\lambda, \theta, -\omega) f(\omega)\ d\omega \end{aligned}$$

which proves the theorem.

Scattering matrix action on L^2

Proof of Lemma

Recall that

$$u(x, \lambda, \omega) = -R_V(\lambda) V e^{-i\lambda\langle \bullet, \omega\rangle}(x)$$

is the outgoing part of the eigenfunction w. Recall also that

$$R_V(\lambda)\rho = R_0(\lambda)\rho(1 + VR_0(\lambda)\rho)^{-1},$$

and $V = \rho V$, so $u = R_0(\lambda) f$ where

$$f(x) = -(1 + VR_0(\lambda)\rho)^{-1} V e^{-i\lambda\langle \bullet, \omega\rangle}(x).$$

Now recall from Haoren's talk:

Sublemma (Asymptotics of the outgoing part)

Let $n \geq 3$ odd, f a Schwartz function on $\mathbb{R}^n$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Then

$$R_0(\lambda) f(r\theta) = e^{i\lambda r} r^{-\frac{n-1}{2}} \left(\frac{1}{4\pi} \left(\frac{\lambda}{2\pi i} \right)^{(n-3)/2} \hat{f}(\lambda\theta) + O(r^{-1}) \right).$$

Scattering matrix action on L^2

Proof of Lemma

Assume without loss of generality that f is Schwartz. Since $u = R_0(\lambda)f$ and

$$u(r\theta, \lambda, \omega) = c_n^-(\lambda r)^{-\frac{n-1}{2}} e^{i\lambda r} b(\lambda, \theta, \omega) + O(r^{-\frac{n+1}{2}}),$$

and the sublemma gives a formula for the highest-order term b of u,

$$\begin{aligned} b(\lambda, \theta, \omega) &= -\frac{1}{4\pi c_n^-} \frac{\lambda^{n-2}}{(2\pi i)^{\frac{n-3}{2}}} \hat{f}(\lambda\theta) \\ &= \frac{\lambda^{n-2}}{(2\pi)^{n-1}(2i)} \int_{\mathbb{R}^n} e^{-i\lambda\langle x, \theta\rangle}(1 + VR_0(\lambda)\rho)^{-1} V e^{-i\lambda\langle \bullet, \omega\rangle}(x)\, dx \end{aligned}$$

so b is the integral kernel of

$$A(\lambda) = a_n \lambda^{n-2} E_\rho(\lambda)(1 + VR_0(\lambda)\rho)^{-1} V E_\rho(\overline{\lambda})^*.$$

This proves the lemma.

Solutions of prescribed incoming part

We now claim that $S(\lambda)$ really is the scattering matrix, in the sense that it maps *any* incoming wave to an outgoing wave.

Theorem

Suppose that $V \in L^\infty_c(\mathbb{R}^n \to \mathbb{R})$, $\lambda \in \mathbb{R} \setminus 0$, $n \geq 3$. *For every* $g \in C^\infty(\mathbb{S}^{n-1})$ *there is a unique* $f \in C^\infty(\mathbb{S}^{n-1})$ *and a unique eigenfunction* $v \in H^2_{loc}(\mathbb{R}^n)$ *such that*

$$(P_V - \lambda^2)v = 0,$$

and

$$v(r\theta) = r^{-\frac{n-1}{2}} \left(e^{i\lambda r} f(\theta) + e^{-i\lambda r} g(\theta)\right) + O(r^{-(n+1)/2}).$$

For such f, g,

$$S(\lambda)g(\theta) = i^{n-1} f(-\theta).$$

So f, g are the outgoing and incoming amplitudes of v, respectively.

Solutions of prescribed incoming part

Proof: Existence

Let $b_n = 1/c_n^+$,

$$u_0(x) = b_n \lambda^{(n-1)/2} \int_{\mathbb{S}^{n-1}} g(\theta) e^{-i\lambda\langle x,\theta\rangle} \, d\theta.$$

By the stationary phase lemma,

$$u_0(r\omega) \sim r^{-(n-1)/2}(e^{-i\lambda r} g(\omega) + e^{-\pi(n-1)i/2} e^{i\lambda r} g(-\omega)).$$

and moreover $\Delta u_0 = \lambda^2 u_0$. Now let $v = u_0 - R_V(\lambda)(V u_0)$. Then

$$P_V v = P_V u_0 - P_V R_V(\lambda)(V u_0) = V u_0 - V u_0 + \lambda^2 v.$$

Therefore v is an eigenfunction. The definition of S implies that S maps g to the outgoing part of v.

Recall from James' talk:

Lemma (Rellich's theorem for the Sommerfeld radiation condition)

If $P = -\Delta$ close to infinity, $Pu = \lambda^2 u$, $\lambda > 0$, $u \in H^2_{loc}$, and

$$(\partial_r - i\lambda)u(r\omega) = O(r^{-(n+1)/2}),$$

then $u = 0$ close to infinity.

If v, v' meet the criteria of the theorem, then

$$(v - v')(r\omega) = O(r^{-(n+1)/2})$$

and this remains true when we apply the operator $\partial_r - i\lambda$ to $v - v'$. So $v - v' = 0$.

Unitarity of scattering matrix

Theorem

Let $V \in L^\infty_c(\mathbb{R}^n \to \mathbb{C})$*,* $n \geq 3$ *odd. Let* $Jf(\theta) = f(-\theta)$*,* $\lambda \in \mathbb{C}$*. Then:*

- ✓ S *is a meromorphic family of operators on* $\mathbb{C}$ *with poles of finite rank.*
- ✓ *There are finitely many poles in the closed upper-half plane.*
- ✓ *If* λ *is a pole and* $\operatorname{Im} \lambda > 0$*, then* $\lambda^2 \in \operatorname{Spec} P_V$*.*
- ✓ $S(\lambda)^{-1} = JS(\lambda)J$*.*
- ✓ *If* V *is real, then* $S(\lambda)^{-1} = S(\overline{\lambda})^*$*.*
- ✓ *If* V *is real, then* S *is an analytic family of unitary operators on* $\mathbb{R}$*.*

The interpretation is that the poles of S are resonances of V and (if V is real, i.e. the Hamiltonian P_V is observable) then if an observer at infinity observes a particle hit the edge of $\operatorname{supp} V$ (and hence the observer forces a wavefunction collapse),

$$\Pr(\text{particle reflects}) + \Pr(\text{particle passes through}) = 1.$$

First three claims are easy! Recall that

$$S(\lambda) = 1 + a_n \lambda^{n-1} E_\rho(\lambda)(1 + VR_0(\lambda)\rho)^{-1} V E_\rho(\overline{\lambda})^*.$$

Since $(1 + VR_0(\lambda)\rho)^{-1}$ is a factor of the meromorphic family of operators $R_V(\lambda)$ and $E_\rho(\lambda)$ is a holomorphic family of operators (hence so is $E_\rho(\overline{\lambda})^*$), S is a meromorphic family of operators. The criteria on poles of positive imaginary part follow from facts about R_V.

Unitarity of scattering matrix

Proof that $S^{-1} = JSJ$

Since S_{abs} maps incoming amplitudes (those of frequency λ) to outgoing amplitudes (those of frequency $-\lambda$), we clearly have $S_{abs}(\lambda)^{-1} = S_{abs}(-\lambda)$, say if $\lambda > 0$ (and hence for any λ by meromorphic continuation). Since

$$S(\lambda) = i^{n-1} S_{abs}(\lambda) J$$

and n is odd,

$$S(-\lambda) = i^{n-1} S_{abs}(\lambda)^{-1} J = JS(\lambda)^{-1} J.$$

Assume that V is real. By meromorphy it suffices to show that if $\lambda \in \mathbb{R}$ then $S(\lambda)$ is unitary. (In particular, $||S(\lambda)||_{L^2 \to L^2} = 1$, so by continuity there are no poles close to λ.) Recall from James' talk:

Lemma (boundary pairing)

Let P be a self-adjoint operator such that $P = -\Delta$ at infinity. Let $u_\ell \in H^2_{loc}$, $(P - \lambda^2)u_\ell = F_\ell$, g_ℓ the incoming amplitude of u_ℓ, f_ℓ the outgoing amplitude of u_ℓ. Then

$$2i\lambda \int_{\mathbb{S}^{n-1}} g_1 \overline{g}_2 - f_1 \overline{f}_2 = \int_{\mathbb{R}^n} F_1 \overline{u}_2 - u_1 \overline{F}_2.$$

We apply the boundary pairing lemma with $P = P_V$, $f = S_{abs}(\lambda)g$, $F = 0$, $u = u_1 = u_2$ to see that $2i\lambda(||g||^2_{L^2} - ||f||^2_{L^2}) = 0$ and conclude that $S_{abs}(\lambda)$ is unitary. This proves the theorem.

Scattering matrix of a black box Hamiltonian

We now consider a black box Hamiltonian. That is, we do not assume that we know V, only the Hamiltonian $P = P_V$.

Definition (preliminary)

An unbounded self-adjoint operator P acting on a dense subspace of $L^2(\mathbb{R}^n)$ is called a *black box Hamiltonian* if there is a compact set $K \subset \mathbb{R}^n$, the *support* of P, such that $1_K(P+i)^{-1}$ is compact, and for every $u \in H^2_c(\mathbb{R}^n \setminus K)$,

$$Pu = -\Delta u.$$

From this we can define the resolvent $R(\lambda) = (P - \lambda^2)^{-1}$.

Scattering matrix of a black box Hamiltonian

Theorem

Suppose that $P = P_V$ *for some* $V \in L^\infty_c(\mathbb{R}^n)$, $n \geq 3$ *odd. Assume that* χ_1, χ_2 *are cutoffs,* V *supported in* $\{\chi_1 = 1\}$, χ_1 *supported in* $\{\chi_2 = 1\}$, χ_2 *supported in* $\{\rho = 1\}$. *Then*

$$S(\lambda) = 1 + a_n \lambda^{n-2} E_\rho(\lambda)[\Delta, \chi_1] R_V(\lambda)[\Delta, \chi_2] E_\rho(\overline{\lambda})^*.$$

One can check that the choice of cutoffs doesn't matter. Thus, if P is a black box Hamiltonian, we may take this formula as the *definition* of the scattering matrix of P, provided that $n \geq 3$ is odd, and the support of P is contained in $\{\chi_1 = 1\}$.

Scattering matrix of a black box Hamiltonian

Proof

By meromorphy we may assume that $\lambda \in \mathbb{R} \setminus 0$. Let

$$E(\lambda, x, \omega) = e^{-i\lambda\langle x,\omega\rangle}$$

be an integral operator $L^2_c(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$. Let $h_1, h_2 \in C^\infty(\mathbb{S}^{n-1})$ be given. Let

$$\begin{aligned} u_1 &= ((1-\chi_2)E(\lambda)^* - R_V(\lambda)[\Delta, \chi_2]E_\rho(\lambda)^*)h_1, \\ u_2 &= (1-\chi_1)E(\lambda)^* h_2. \end{aligned}$$

Let $F_j = (P_V - \lambda^2)u_j$. We claim

$$\begin{aligned} F_1 &= 0, \\ F_2 &= [\Delta, \chi_1]E_\rho(\lambda)^* h_2. \end{aligned}$$

Scattering matrix of a black box Hamiltonian

Proof: Computation of F

First observe that since $V = 0$ when $\chi_j \neq 1$,

$$\begin{aligned}(P_V - \lambda^2)(1-\chi_j)E(\lambda)^* &= (-\Delta - \lambda^2)(1-\chi_j)E(\lambda)^* \\ &= (1-\chi_j)(-\Delta-\lambda^2)E(\lambda)^* + [\Delta, \chi_j]E(\lambda)^*.\end{aligned}$$

Since $E(\lambda)^*$ returns eigenfunctions of Δ at frequency λ and $\rho(x) = 1$ for all relevant x, this implies

$$(P_V - \lambda^2)(1-\chi_j)E(\lambda)^* = [\Delta, \chi_j]E_\rho(\lambda)^*.$$

This immediately implies the claim for F_2, and since $V = 0$ when $\chi_2 \neq 1$,

$$\begin{aligned}F_1 &= ([\Delta, \chi_2]E_\rho(\lambda)^* - R_V(\lambda)[\Delta, \chi_2]E_\rho(\lambda)^*)h_1 \\ &= (1 - R_0(\lambda))[\Delta, \chi_2]E_\rho(\lambda)^* h_1 = 0.\end{aligned}$$

The last line follows because $[\Delta, \chi_2]E(\lambda)^*$ returns eigenfunctions at frequency λ, so $R_0(\lambda)$ is the identity.

Scattering matrix of a black box Hamiltonian

Proof: Computation of boundary pairing

Recall $(1-\chi_2)[\Delta,\chi_1]=0$ (since $\chi_2=1$ on $\operatorname{supp}\chi_1$) and $[\Delta,\chi_1]^*=-[\Delta,\chi_1]$ (since Δ,χ_1 are self-adjoint). It follows that

$$\begin{aligned} u_1\overline{F}_2 &= ((1-\chi_2)E(\lambda)^*h_1 - R_V(\lambda)[\Delta,\chi_2]E_\rho(\lambda)^*h_1)([\Delta,\chi_1]E_\rho(\lambda)^*\overline{h}_2) \\ &= -(R_V(\lambda)[\Delta,\chi_2]E_\rho(\lambda)^*h_1)([\Delta,\chi_1]E_\rho(\lambda)^*\overline{h}_2). \end{aligned}$$

Let $\langle\cdot,\cdot\rangle_X$ be the inner product of $L^2(X)$. Then

$$\begin{aligned} -\langle u_1,F_2\rangle_{\mathbb{R}^n} &= \langle R_V(\lambda)[\Delta,\chi_2]E_\rho(\lambda)^*h_1,[\Delta,\chi_1]E_\rho(\lambda)^*h_2\rangle_{\mathbb{R}^n} \\ &= -\langle E_\rho(\lambda)[\Delta,\chi_1]R_V(\lambda)[\Delta,\chi_2]E_\rho(\lambda)^*h_1,h_2\rangle_{\mathbb{S}^{n-1}} \\ &= \langle G(\lambda)h_1,h_2\rangle_{\mathbb{S}^{n-1}} \end{aligned}$$

where $G(\lambda)=E_\rho(\lambda)[\Delta,\chi_1]R_V(\lambda)[\Delta,\chi_2]E_\rho(\lambda)^*$.

Now

$$u_1 = ((1-\chi_2)E(\lambda)^* - R_V(\lambda)[\Delta, \chi_2]E_\rho(\lambda)^*)h_1$$

and R_V is the outgoing resolvent, so the incoming part of u_1 is equal to the incoming part of $(1-\chi_2)E(\lambda)^* h_1$, which is equal to the incoming part of

$$E(\lambda)^* h_1(x) = \int_{\mathbb{S}^{n-1}} e^{i\lambda\langle x,\omega\rangle} h_1(\omega)\ d\omega.$$

Applying the stationary phase lemma, we see that the incoming part of $E(\lambda)^* h_1$ is

$$g_1(\theta) = c_n^- \lambda^{-(n-1)/2} h_1(-\theta).$$

Therefore, by definition of $S(\lambda)$, the outgoing part of u_1 is

$$f_1(\theta) = c_n^- i^{1-n} \lambda^{-(n-1)/2} S(\lambda) h_1(\theta).$$

Scattering matrix of a black box Hamiltonian

Proof: Decomposition of u_2

Since

$$u_2 = E(\lambda)^* h_2$$

close to infinity, we can apply the stationary phase lemma to $E(\lambda)^* h_2$ to see that

$$g_2(\theta) = c_n^- \lambda^{-(n-1)/2} h_2(-\theta)$$

and

$$f_2(\theta) = c_n^- \lambda^{-(n-1)/2} i^{1-n} h_2(\theta).$$

Notice that we did not use the scattering matrix to recover f_2 from g_2. This will be important when every term cancels out except $S(\lambda)$ and the terms that we want.

Scattering matrix of a black box Hamiltonian

Proof: Comparing boundary pairings

By the boundary pairing lemma,

$$\begin{aligned}\int_{\mathbb{R}^n} F_1\overline{u}_2 - u_1\overline{F}_2 &= 2i\lambda(\langle g_1, g_2\rangle_{\mathbb{S}^{n-1}} - \langle f_1, f_2\rangle_{\mathbb{S}^{n-1}}) \\ &= \langle 2i\lambda^{2-n}(2\pi)^{n-1}(1 - S(\lambda))h_1, h_2\rangle_{\mathbb{S}^{n-1}}.\end{aligned}$$

On the other hand, $F_1 = 0$, so

$$\int_{\mathbb{R}^n} F_1\overline{u}_2 - u_1\overline{F}_2 = -\langle u_1, F_2\rangle_{\mathbb{R}^n} = \langle G(\lambda)h_1, h_2\rangle_{\mathbb{S}^{n-1}}.$$

Since h_1, h_2 were arbitrary we must have

$$2i\lambda^{2-n}(2\pi)^{n-1}(1 - S(\lambda)) = G(\lambda).$$

Since $G(\lambda) = E_\rho(\lambda)[\Delta, \chi_1]R_V(\lambda)[\Delta, \chi_2]E_\rho(\lambda)^*$, we can solve for $S(\lambda)$ to complete the proof.

Multiplicity of poles

Recall that $m_R(\lambda)$ is the multiplicity of the resonance λ as a pole of the scattering resolvent. There is another way to define resonance multiplicity: Let

$$m_S(\lambda) = -\frac{1}{2\pi i} \operatorname{tr} \oint_\lambda S(\zeta)^{-1} \partial_\zeta S(\zeta)\, d\zeta.$$

To interpret this quantity we may use $(\log \det S)' = \operatorname{tr}(S^{-1} S')$, and recall the argument principle: so we are really counting the zeroes and poles of $\det S$.

Theorem

Let $V \in L^\infty_c(\mathbb{R}^n)$, $n \geq 3$ odd. Then $m_S(\lambda) = m_R(\lambda) - m_R(-\lambda)$.

Multiplicity of poles

Proof: Reduction to nonzero simple poles

Lemma

Without loss of generality, we may assume that R_V only has simple poles.

The idea behind the proof of the lemma is that the zeroes and poles of $\det S(\lambda)$ depend continuously on V in compact sets, because if $||V - V'||_{L^2 \to L^2}$ is small then $1 + VR_0(\lambda)\rho$ is invertible iff $1 + V'R_0(\lambda)\rho$ is invertible. We omit the details.
Regardless of what $m_R(0)$ is, $m_R(0) - m_R(-0) = 0$. But the formula for a black box Hamiltonian's scattering matrix says that

$$S(0) = 1 + a_n 0^{n-2} E_\rho(0)[\Delta, \chi_1] R_V(0)[\Delta, \chi_2] E_\rho(0)^* = 1,$$

so $m_S(0) = 0$.
If R_V is holomorphic at λ then $1 + VR_0(\lambda)\rho$ is invertible, so $m_S(\lambda) = 0$.
So we may assume that $\lambda \neq 0$ is a simple pole of R_V, and must prove that $m_R(\lambda) - m_R(-\lambda) = m_S(\lambda)$.

Multiplicity of poles

Proof: Residue of the scattering matrix

Recall from Haoren's talk:

Lemma (singular part of the resolvent, simple case)

If $m_R(\lambda) = 1$ and $\lambda \neq 0$ then there is an eigenfunction $u \in H^2_{loc}$ such that $P_V u = \lambda^2 u$ and $\mathrm{Res}(R_V, \lambda) = u \otimes u$.

Let $U_j(\theta) = E_\rho(\lambda)[\Delta, \chi_j]u$. By the formula for the scattering matrix of a black box Hamiltonian, modulo holomorphic terms,

$$\begin{aligned} S(\zeta) &= a_n \lambda^{n-2} E_\rho(\lambda)[\Delta, \chi_1] \frac{u \otimes u}{\lambda - \zeta} [\Delta, \chi_2] E_\rho(\overline{\lambda})^* \\ &= \frac{a_n \lambda^{n-2}}{\lambda - \zeta} E_\rho(\lambda)[\Delta, \chi_1] u \otimes [\Delta, \chi_2]^* E_\rho(\overline{\lambda})^* u \\ &= -\frac{a_n \lambda^{n-2} U_1 \otimes U_2}{\lambda - \zeta} \end{aligned}$$

so $\mathrm{Res}(S, \lambda) = -U_1 \otimes U_2$.

Multiplicity of poles

Proof: Fourier analytic computations

Lemma

Let $U(\theta) = \widehat{Vu}(\theta\lambda)$. Then $U = U_1 = U_2$. Moreover, $U \neq 0$.

Since u is an eigenfunction, $[\Delta, \chi_j]u = \Delta\chi_j u - \chi_j \Delta u = (\Delta + \lambda^2)\chi_j u + Vu$. But

$$E_\rho(\lambda)[\Delta, \chi_j]u(\theta) = \int_{\mathbb{R}^n} e^{-i\lambda\langle\theta,x\rangle}((\Delta+\lambda^2)(\chi_j u)(x) + V(x)u(x))\,dx$$
$$= (\lambda^2 - |\theta\lambda|^2)\widehat{\chi_j u}(\theta\lambda) + \widehat{Vu}(\theta\lambda) = U(\theta).$$

This proves $U_1 = U_2 = U$. If U is identically 0 then we proceed as in the proof of Rellich's theorem. First we use Paley-Wiener theory to show that u has compact support. We then use Carleman estimates to show that $u = 0$, a contradiction. We omit the details. This proves the lemma.

Multiplicity of poles

Proof: Gohberg-Segal factorization

Lemma (Theorem C.10)

Let A be a meromorphic family of Fredholm operators on a Riemann surface Ω, and let $\mu \in \Omega$. If the Fredholm index of the holomorphic part of A is 0 near μ, then there are holomorphic families of invertible operators U, V near μ, finitely many operators P_m such that if $m \neq 0$ then $\operatorname{rank} P_m \leq 1$, *and that*

$$A(\lambda) = U(\lambda)(P_0 + \sum_{m \neq 0} (\lambda - \mu)^m P_m) V(\lambda).$$

By the Fourier analysis lemma, there is a simple pole of S at λ. So by Theorem C.10, there are $\delta > 0$ and operators such that

$$S(\zeta) = G(\zeta)(Q_{-1}(\lambda - \zeta)^{-1} + Q_0 + Q_1(\lambda - \zeta) + \cdots)F(\zeta), \qquad |\lambda - \zeta| < \delta$$
$$S(-\zeta) = \widetilde{G}(\zeta)(\widetilde{Q}_{-1}(\lambda - \zeta)^{-1} + \widetilde{Q}_0 + \widetilde{Q}_1(\lambda - \zeta) + \cdots)\widetilde{F}(\zeta), \qquad |\lambda + \zeta| < \delta.$$

Multiplicity of poles

Proof: Comparing ranks of Gohberg-Segal factors

We proved that

$$\begin{aligned} S(\zeta) &= G(\zeta)(Q_{-1}(\lambda-\zeta)^{-1} + Q_0 + Q_1(\lambda-\zeta) + \cdots)F(\zeta), \qquad |\lambda-\zeta| < \delta \\ S(-\zeta) &= \widetilde{G}(\zeta)(\widetilde{Q}_{-1}(\lambda-\zeta)^{-1} + \widetilde{Q}_0 + \widetilde{Q}_1(\lambda-\zeta) + \cdots)\widetilde{F}(\zeta), \qquad |\lambda+\zeta| < \delta. \end{aligned}$$

In fact $\operatorname{rank} \widetilde{Q}_{-1} = 1$ iff $-\lambda$ is a simple pole, and $\operatorname{rank} \widetilde{Q}_{-1} = 0$ otherwise; that is, $\operatorname{rank} \widetilde{Q}_{-1} = m_R(-\lambda)$. We already know that $\operatorname{rank} Q_{-1} = 1 = m_R(\lambda)$. Moreover,

$$\begin{aligned} S(-\zeta) &= JS(\zeta)^{-1}J \\ &= JF(\zeta)^{-1}(Q_{-1}(\lambda-\zeta) + Q_0 + Q_1(\lambda-\zeta)^{-1} + \cdots)G(\zeta)^{-1}J. \end{aligned}$$

Comparing like terms we see that $Q_{-1} = \widetilde{Q}_1$ and $\widetilde{Q}_{-1} = Q_1$. So

$$m_S(\lambda) = \operatorname{rank} Q_{-1} - \operatorname{rank} Q_1 = \operatorname{rank} Q_{-1} - \operatorname{rank} \widetilde{Q}_{-1} = m_R(\lambda) - m_R(-\lambda).$$

This proves the theorem.